Week 10

9 Suppose $u, v \in V$ and $\|u\| \leq 1$ and $\|v\| \leq 1$. Prove that

$$\sqrt{1-\|u\|^2}\sqrt{1-\|v\|^2} \leq 1-|\langle u, v\rangle|.$$

Sol^n Recall Cauchy-Schwarz inequality $\|u\|\|v\| \geq |\langle u,v\rangle|$

Note that $\sqrt{1-\|u\|^2}\sqrt{1-\|v\|^2} \geq 0$ and $1-|\langle u,v\rangle| \geq 1-\|u\|\|v\| \geq 0$

It suffices to prove $(1-\|u\|\|v\|)^2 - (\sqrt{1-\|u\|^2}\sqrt{1-\|v\|^2})^2 \geq 0$

$(1-\|u\|\|v\|)^2 - (1-\|u\|^2)(1-\|v\|^2)$

$= 1 - 2\|u\|\|v\| + (\|u\|\|v\|)^2 - 1 + \|u\|^2 + \|v\|^2 - \|u\|^2\|v\|^2$

$= \|u\|^2 - 2\|u\|\|v\| + \|v\|^2 = (\|u\|-\|v\|)^2 \geq 0$

Hence $1-|\langle u,v\rangle| \geq 1-\|u\|\|v\| \geq \sqrt{1-\|u\|^2}\sqrt{1-\|v\|^2}$

Simple fact: If $a, b \geq 0$ then $a \geq b \iff a^2 - b^2 \geq 0$

Geometric Interpretation

For $V = \mathbb{R}^3$, $\langle\, ,\, \rangle$: the standard inner product, u, v vectors lying on xy-plane, (e_1, e_2, e_3) standard orthonormal basis of $\mathbb{R}^3$

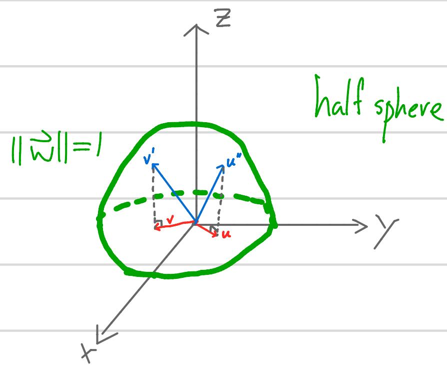
$u' := \lambda u + \sqrt{1-\|u\|^2}\cdot e_3$ where $\lambda = \begin{cases} 1 & \text{if } \langle u,v\rangle = 0 \\ \frac{|\langle u,v\rangle|}{\langle u,v\rangle} & \text{if } \langle u,v\rangle \neq 0 \end{cases}$ $\quad v' := v + \sqrt{1-\|v\|^2}\, e_3$

Note that $\langle u, e_3\rangle = \langle v, e_3\rangle = 0$, $|\lambda| = 1$

Then $\|u'\|^2 = \langle \lambda u + \sqrt{1-\|u\|^2}\cdot e_3, \lambda u + \sqrt{1-\|u\|^2}\cdot e_3\rangle$

$= \|\lambda u\|^2 + \|\sqrt{1-\|u\|^2}\cdot e_3\|^2$

$= |\lambda|^2\|u\|^2 + (1-\|u\|^2)\cdot 1 = 1$



Similarly $\|v'\|^2 = 1$

By C. S. ineq.

$1 = \|u'\|\|v'\| \geq |\langle u', v'\rangle| = |\langle \lambda u + \sqrt{1-\|u\|^2}e_3, v + \sqrt{1-\|v\|^2}\, e_3\rangle|$

$= |\langle \lambda u, v\rangle + \langle \sqrt{1-\|u\|^2}e_3, \sqrt{1-\|v\|^2}e_3\rangle| = |\lambda\langle u,v\rangle + \sqrt{1-\|u\|^2}\sqrt{1-\|v\|^2}|$

$= |\,|\langle u,v\rangle| + \sqrt{1-\|u\|^2}\sqrt{1-\|v\|^2}\,| = |\langle u,v\rangle| + \sqrt{1-\|u\|^2}\sqrt{1-\|v\|^2}$

$\therefore 1 - |\langle u,v\rangle| \geq \sqrt{1-\|u\|^2}\sqrt{1-\|v\|^2}$

Q2 Let $V = P_2(\mathbb{R})$. Define an inner product on V by
$\langle f(x), g(x)\rangle = \int_{-1}^{1} f(x)g(x)(1-x^2)\,dx$ for all $f(x), g(x) \in V$.
Apply Gram-Schmidt process on the basis $\beta = (\overset{=v_1}{1}, \overset{=v_2}{x}, \overset{=v_3}{x^2})$
to get an O.N.B. of V

Sol$^{\underline{n}}$ Let $u_1 = v_1 = 1$ $\quad \|u_1\|^2 = \int_{-1}^{1} 1^2 \cdot (1-x^2)\,dx = \left[x - \frac{x^3}{3}\right]_{-1}^{1} = \frac{4}{3}$

$\therefore e_1 = \frac{1}{\|u_1\|} u_1 = \sqrt{\frac{3}{4}}$

Let $u_2 = v_2 - \frac{\langle v_2, u_1\rangle}{\langle u_1, u_1\rangle} u_1$ $\quad \langle v_2, u_1\rangle = \int_{-1}^{1} x \cdot 1 \cdot (1-x^2)\,dx = 0$

$\therefore u_2 = v_2 - 0 = x$ $\quad \|u_2\|^2 = \int_{-1}^{1} x^2 (1-x^2)\,dx = \left[\frac{x^3}{3} - \frac{x^5}{5}\right]_{-1}^{1} = \frac{4}{15}$

$e_2 = \frac{1}{\|u_2\|} u_2 = \sqrt{\frac{15}{4}}\, x$

Let $u_3 = v_3 - \frac{\langle v_3, u_1\rangle}{\langle u_1, u_1\rangle} u_1 - \frac{\langle v_3, u_2\rangle}{\langle u_2, u_2\rangle} u_2$

$\langle v_3, u_1\rangle = \int_{-1}^{1} x^2 \cdot 1\,(1-x^2)\,dx = \frac{4}{15}$

$\langle v_3, u_2\rangle = \int_{-1}^{1} x^2 \cdot x(1-x^2)\,dx = 0$

$\int_{-1}^{1} x^4 (1-x^2)\,dx = \frac{4}{35}$

$u_3 = x^2 - (4/15)/(4/3) \cdot 1 - 0$

$= x^2 - \frac{1}{5}$

$\|u_3\|^2 = \int_{-1}^{1} \left(x^2 - \frac{1}{5}\right)^2 (1-x^2)\,dx$

$= \int_{-1}^{1} \left(x^4 - \frac{2}{5}x + \frac{1}{25}\right)(1-x^2)\,dx$

$= \frac{4}{35} - \frac{2}{5}\left(\frac{4}{15}\right) + \frac{1}{25}\left(\frac{4}{3}\right) = \frac{32}{525}$

$\therefore e_3 = \frac{1}{\|u_3\|} u_3 = \sqrt{\frac{525}{32}}\left(x^2 - \frac{1}{5}\right)$.

The required O.N.B. is $\left(\sqrt{\frac{3}{4}}, \sqrt{\frac{15}{4}}\, x, \sqrt{\frac{525}{32}}\left(x^2 - \frac{1}{5}\right)\right)$

11 Suppose $\langle\cdot,\cdot\rangle_1$ and $\langle\cdot,\cdot\rangle_2$ are inner products on V such that $\langle v,w\rangle_1 = 0$ if and only if $\langle v,w\rangle_2 = 0$. Prove that there is a positive number c such that $\langle v,w\rangle_1 = c\langle v,w\rangle_2$ for every $v,w \in V$.

Sol^n If $V=\{0\}$ the trivial vector space, any inner product on V is just the zero function. So we may take c to be any the real no.
So we may assume V is nontrivial. Pick $v_0 \in V$ $v_0 \neq 0$.
Then $\langle v_0,v_0\rangle_2 \neq 0$ by IPS. Take $c = \frac{\langle v_0,v_0\rangle_1}{\langle v_0,v_0\rangle_2} > 0$
Define a function $f: V \to \mathbb{F}$ by $f(v) = \frac{\langle v,v\rangle_1}{\langle v,v\rangle_2}$ if $v\neq 0$ and $f(0)=c$
Let $v,w\in V$. If $v=0$ or $w=0$ then $\langle v,w\rangle_1 = 0 = c\langle v,w\rangle_2$
So assume $v\neq 0$ $w \neq 0$. If $\langle v,w\rangle_2 \neq 0$ then $\langle v,w\rangle_1 \neq 0$.
Since $\langle v - \frac{\langle v,w\rangle_2}{\langle w,w\rangle_2} w, w\rangle_2 = 0$ $\langle v - \frac{\langle v,w\rangle_2}{\langle w,w\rangle_2} w, w\rangle_1 = 0$
$\therefore \langle v,w\rangle_1 = \frac{\langle v,w\rangle_2}{\langle w,w\rangle_2}\langle w,w\rangle_1 = \frac{\langle w,w\rangle_1}{\langle w,w\rangle_2}\langle v,w\rangle_2 = f(w)\langle v,w\rangle_2$ —— $(\star)$

We want to show that f is a constant function

$\frac{\langle v,w\rangle_1}{\langle v,w\rangle_2} = \frac{\langle w,w\rangle_1}{\langle w,w\rangle_2}$ Take conjugation on both sides, since R.H.S. is a real no. $\frac{\langle w,v\rangle_1}{\langle w,v\rangle_2} = f(w)$ In particular $\frac{\langle v,w\rangle_1}{\langle v,w\rangle_2} = f(v)$ too.

If $\langle v,w\rangle_2 = 0$, $\langle v,w\rangle_1 = 0$ let $u = v+w$ Then $\langle u,v\rangle_2 = \langle v,v\rangle_2 + \langle w,v\rangle_2 = \langle v,v\rangle_2 > 0$
Similarly $\langle u, w\rangle_2 > 0$ Therefore $f(v) = \frac{\langle u,v\rangle_1}{\langle u,v\rangle_2} = f(u) = \frac{\langle u,w\rangle_1}{\langle u,w\rangle_2} = f(w)$
In particular, $f(v_0) = f(w)$ $\forall w \in V$.
$\therefore \langle v,w\rangle_1 = c\langle v,w\rangle_2$ $\forall v,w\in V$.